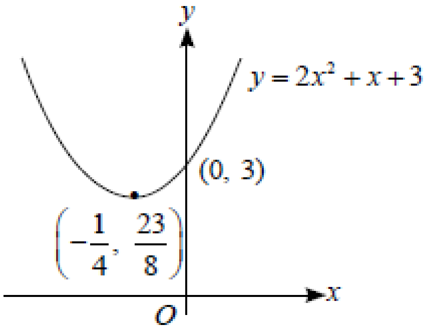
Exercise 1

A Showing Functions exist

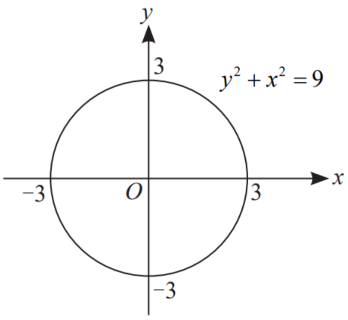
1
Solution

(a)



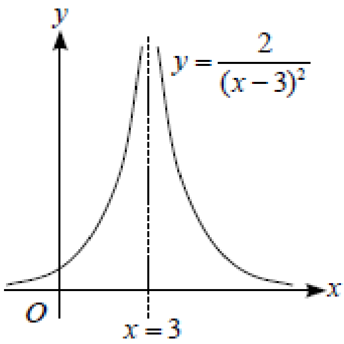
From the above diagram, for any vertical line $x = b,\ b \in \mathbb{R}$, cuts the curve **exactly at one point**.
Therefore, the curve $y = 2x^2 + x + 3$ is a function.

(b) Solution



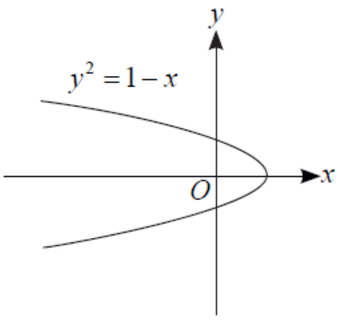
From, the above diagram, for any vertical line, $x = b$, where $-3 < b < 3$, **cuts the curve twice**.
Therefore, the curve $y^2 + x^2 = 9$ is not a function.

(c) Solution



Since any vertical line (except $x = 3$) cuts the graph once, one x-value corresponds to one y-value, so this is a function. Therefore, the curve $y = \dfrac{2}{(x-3)^2}$ is a function.

(d) Solution



Method 1

Since $x = 0 \Rightarrow y = -1$ or 1. $\therefore$ this is not a function.

Method 2

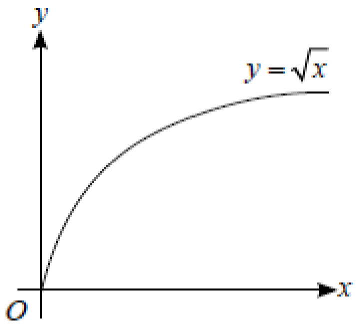
Since $x = 0$ cuts the graph twice, $y^2 = \sqrt{1-x}$ is not a function.

Exercise 1

B Domain of a Function

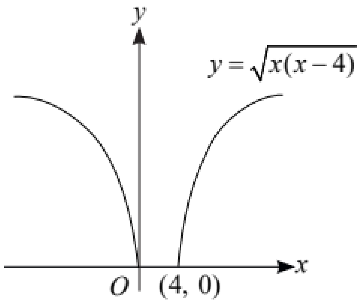
2
Solution

(a)



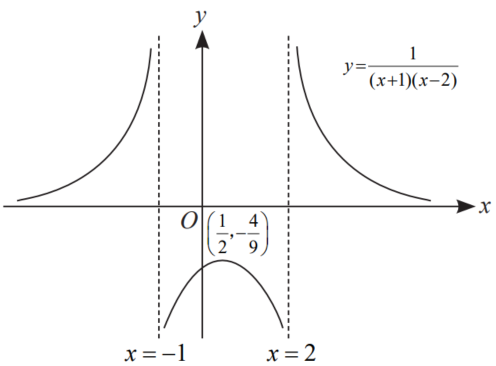
The largest possible domain is $x \geq 0$.

(b) Solution



The largest possible domain for the function is $x \leq 0$ and $x \geq 4$

(c) Solution

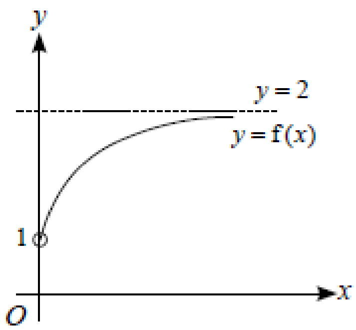


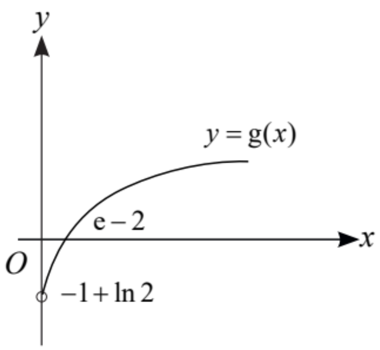
The largest possible domain for the function is $x \in \mathbb{R}$, except -1 and 2.

Exercise 1

C Range of a Function

3
(a) Solution



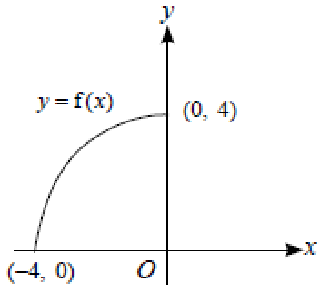


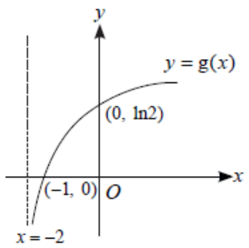
(b)

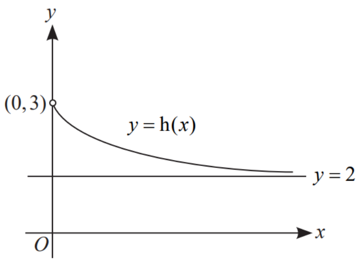
The range of $f = (1, 2)$.

The range of $g = (\ln 2 - 1, \infty)$.

4
(a) Solution



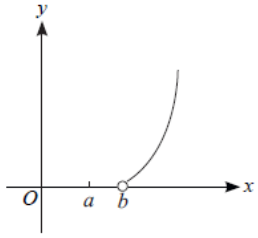




(b)

The range of function f is $R_f = [0,\ 4]$.
The range of function g is $R_g = \mathbb{R}$.
The range of function g is $R_h = (2,\ 3)$.

5
(a) Solution



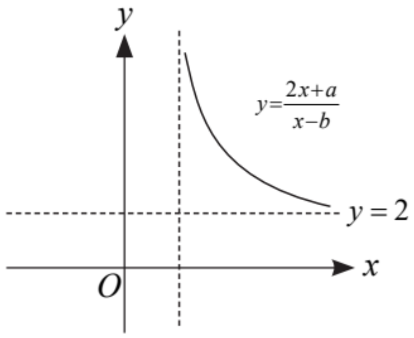
Sketch the graph $f: x \mapsto (x-a)(x-b), x \in \mathbb{R}, x > b$

From the graph of f, the range of f is $(0, \infty)$.

(b)

The graph of $y = \dfrac{2x+a}{x-b} = 2 + \dfrac{a+2b}{x-b}$ has vertical asymptote $x = b$ and horizontal asymptote $y = 2$.
For the domain $x > b$, $R_g = (2, \infty)$.

Alternative, we can sketch the graph $y = \dfrac{2x+a}{x-b}$ to determine its range.

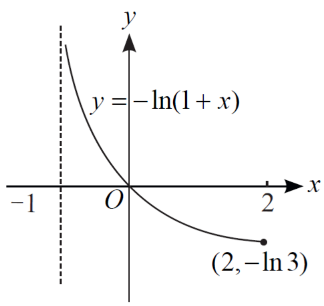


From the diagram, $R_g = (2, \infty)$.

Exercise 1

D One - one Functions and Inverse Functions

6
Solution
(a)



From the graph of $y = -\ln(1+x)$, any horizontal line $y = k$, $k \in \mathbb{R}_{\mathrm{f}}$, cuts the curve exactly once, so the function f is one - one.

(b) Let $y = -\ln(1+x)$

$$-y = \ln(1+x)$$
$$1+x = \mathrm{e}^{-y}$$
$$x = \mathrm{e}^{-y} - 1$$

$\therefore\ \mathrm{f}^{-1}(x) = \mathrm{e}^{-x} - 1$

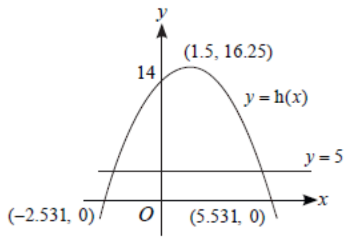
$\mathrm{D}_{\mathrm{f}^{-1}} = \mathbb{R}$

(c) $\mathrm{f}^{-1}(x) = 1$

$$x = \mathrm{f}(1)$$
$$x = -\ln(1+1)$$
$$x = -\ln 2$$

\

7
Solution
(a)



From the diagram above, we note that the line $y = 5$ cuts the graph of h at 2 points.
Therefore, h is not 1-1 function.

(b) From the graph, we can see that for function h to be one-one function, the greatest value k must be $\frac{3}{2}$.

$\therefore$ the greatest value $k = \frac{3}{2}$.

Let $y = \mathrm{h}(x)$.

$y = -x^2 + 3x + 14$

By means of completing the square,

$$y = -\left(x - \frac{3}{2}\right)^2 + \frac{65}{4} \quad \triangleleft \text{ express } x \text{ in term of } y$$

$$y - \frac{65}{4} = -\left(x - \frac{3}{2}\right)^2$$

$$\frac{65}{4} - y = \left(x - \frac{3}{2}\right)^2$$

$$\therefore \quad x = \frac{3}{2} \pm \sqrt{\frac{65}{4} - y}$$

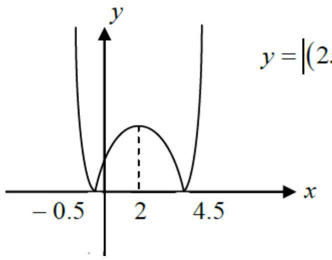
Since $x \le \frac{3}{2}$,

$$x = \frac{3}{2} - \sqrt{\frac{65}{4} - y} \text{ and we reject } x = \frac{3}{2} + \sqrt{\frac{65}{4} - y}$$

$$\therefore \ \mathrm{h}^{-1}(x) = \frac{3}{2} - \sqrt{\frac{65}{4} - x},\ x \in \mathbb{R},\ x \le \frac{65}{4}$$

8
Solution

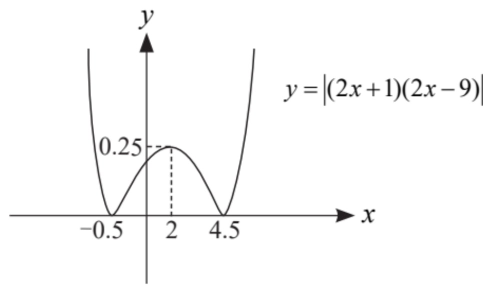
(a) Sketch the graph of function f.



Least value of $a = -0.5$ and greatest value of $b = 2$.

(b) Domain of f^{-1} = Range of f under the restricted domain.

$\therefore\ D_{f^{-1}} = [0, 25]$.



Let $y = f(x)$

$= -(2x+1)(2x-9)$

$y = -\left[4x^2 - 16x - 9\right]$ ◃ completing the square

$y = -4\left[x^2 - 4x\right] + 9$

$y = -4\left[(x-2)^2 - 4\right] + 9$

$y = -4(x-2)^2 + 25$

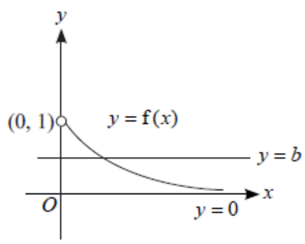
$(x-2)^2 = \dfrac{y-25}{-4}$

$x = 2 \pm \sqrt{\dfrac{25-y}{4}}$

$\therefore x = 2 - \dfrac{\sqrt{25-x}}{2}$ or $x = 2 + \dfrac{\sqrt{25-x}}{2}$ (Rejected since the restricted domain of f is $-0.5 \le x \le 2$)

Hence $f^{-1}(x) = 2 - \dfrac{\sqrt{25-x}}{2}$.

9
Solution
(a)



For any horizontal line $y = b$, where $0 < b < 1$ cuts the graph of f at exactly one point.
Hence, f is one-one function.

Let $\quad y = \mathrm{e}^{-2x}$

$$\ln y = \ln \mathrm{e}^{-2x}$$

$$\ln y = -2x$$

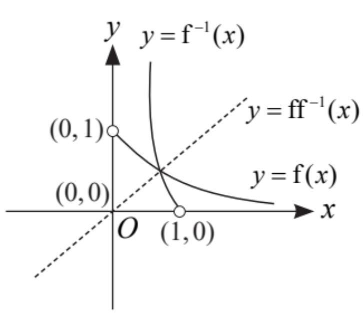
$$x = -\frac{1}{2}\ln y$$

$\mathrm{f}^{-1} : x \mapsto -\frac{1}{2}\ln x, \quad 0 < x < 1$

Range of f $= (0,\ 1)$.
Since Domain of f^{-1} = Range of f
$\therefore$ Domain of $\mathrm{f}^{-1} = (0,\ 1)$

Domain of f $= (0,\ \infty)$.
Since Domain of f = Range of f^{-1}
$\therefore$ Range of $\mathrm{f}^{-1} = (0,\ \infty)$

(b)



The relationship between the f and its inverse is the reflection in the line $y = x$.

Learning Point:
A point (a, b) on the $y = \mathrm{f}(x)$ will become point (b, a) on $y = \mathrm{f}^{-1}(x)$.

(c) To find the point of intersection between f and f^{-1}, we equate $f(x) = x$.

i.e. $e^{-2x} = x$

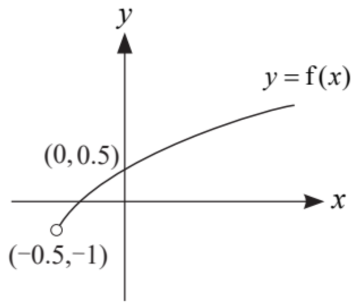
$\ln x = -2x$

$\therefore \ln x - 2x = 0$ (Shown)

(d) Using a graphic calculator to solve equation $\ln x - 2x = 0$.

$\therefore x = 0.4263$

10.
Solution
(a)



(b)

$$y = \mathrm{f}(x)$$

$$y = \sqrt{x+1} - \frac{1}{2}$$

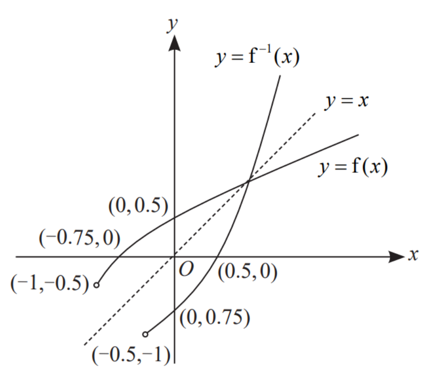
$$y + \frac{1}{2} = \sqrt{x+1}$$

$$x = \left(y + \frac{1}{2}\right)^2 - 1$$

$$\therefore \mathrm{f}^{-1}(x) = \left(x + \frac{1}{2}\right)^2 - 1$$

$\mathrm{D}_{\mathrm{f}^{-1}} = \left(-\frac{1}{2}, \infty\right)$ and $\mathrm{R}_{\mathrm{f}^{-1}} = (-1, \infty)$

(c)



(d) The line is $y = x$

Method 1

Consider $f^{-1}(x) = x$

$$\left(x+\frac{1}{2}\right)^2 - 1 = x$$

$$x^2 + x + \frac{1}{4} - 1 = x$$

$$x^2 = \frac{3}{4}$$

$$x = \frac{3}{4}$$

$x = \dfrac{\sqrt{3}}{2}$ or $-\dfrac{\sqrt{3}}{2}$ (rejected as $x > -0.5$)

Method 2

Consider $f(x) = x$

$$\sqrt{x+1} - \frac{1}{2} = x$$

$$\sqrt{x+1} = x + \frac{1}{2}$$

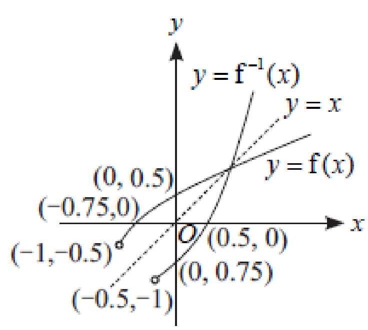
$$x + 1 = x^2 + x + \frac{1}{4}$$

$$x^2 = \frac{3}{4}$$

$x = \dfrac{\sqrt{3}}{2}$ or $-\dfrac{\sqrt{3}}{2}$ (rejected as $x > -0.5$)

(e) From the sketch in **(c)**

The solution set is $\left\{x \in \mathbb{R}, -0.5 < x \le \dfrac{\sqrt{3}}{2}\right\}$.



Exercise 1

E Composite Functions

11
Solution

(a) $\mathrm{fg}(x) = \mathrm{f}\left[\mathrm{g}(x)\right]$

$= \mathrm{f}(x^2 + 4)$

$= (x^2 + 4)^2$

$= x^4 + 8x^2 + 16$

$\mathrm{fg}(x) \mapsto x^4 + 8x^2 + 16, x \in \mathbb{R}$

(b) $\mathrm{gf}(x) = \mathrm{g}\left[\mathrm{f}(x)\right]$

$= \mathrm{g}\left[x^2\right]$

$= (x^2)^2 + 4$

$\mathrm{gf} : x \mapsto x^4 + 4, x \in \mathbb{R}$

(c) $\mathrm{ff}(x) = \mathrm{f}\left[\mathrm{f}(x)\right]$

$= \mathrm{f}\left[x^2\right]$

$= (x^2)^2$

$= x^4$

$\mathrm{f}^2 : x \mapsto x^4, x \in \mathbb{R}$

Learning Point:
Refer to **(a)** and **(b)**. we note that $\mathrm{fg}(x) \neq \mathrm{gf}(x)$.

12
Solution

Let $y = \mathrm{f}(x) = x + 1$

$x = y - 1$ ⊲ express x in terms of y

$\therefore\ \mathrm{f}^{-1}(x) = x - 1$

(a) $x = \mathrm{ff}^{-1}(x)$

$$\mathrm{g}(x) = \mathrm{g}\left[\mathrm{f}(\mathrm{f}^{-1}(x))\right]$$
$$= \mathrm{g}\left[\mathrm{f}(x-1)\right]$$

Given $\mathrm{gf}(x) = x^2$

$\therefore\ \mathrm{g}\left[\mathrm{f}(x-1)\right] = (x-1)^2$ ⊲ replace x by $x - 1$

$\therefore\ \mathrm{g}(x) = (x-1)^2$

(b) $x = \mathrm{f}^{-1}\mathrm{f}(x)$

$$\mathrm{g}(x) = \left[\mathrm{f}^{-1}\mathrm{f}\left(\mathrm{g}(x)\right)\right]$$
$$\mathrm{g}(x) = \mathrm{f}^{-1}(x^2)$$

$\because\ \mathrm{f}^{-1}(x) = x - 1$

$\mathrm{f}^{-1}(x^2) = x^2 - 1$ ⊲ replace x with x^2

$\therefore\ \mathrm{g}(x) = x^2 - 1$

13
Solution

(a) Given $\quad \mathrm{hg}(x) = \mathrm{e}^{-1}(x+2)$

$$\mathrm{h}(-1+\ln(x+2)) = \mathrm{e}^{-1}(x+2)$$

$$= \mathrm{e}^{-1}\mathrm{e}^{\ln(x+2)} \qquad \triangleleft\ \mathrm{e}^{\ln a} = a$$

$$\mathrm{h}(-1+\ln(x+2)) = \mathrm{e}^{-1+\ln(x+2)}$$

$$\mathrm{h}(x) = \mathrm{e}^{x} \qquad \triangleleft \text{ replaced } -1+\ln(x+2) \text{ by } x$$

$\therefore$ the expression for $\mathrm{h}(x)$ is e^{x}.

(b) $\mathrm{fh}(x) = \mathrm{f}\left[\mathrm{h}(x)\right]$

$$= \mathrm{f}\left[\mathrm{e}^{x}\right]$$

$$= \frac{\mathrm{e}^{x}+1}{\mathrm{e}^{x}}$$

$$\mathrm{fh} : x \mapsto \frac{\mathrm{e}^{x}+1}{\mathrm{e}^{x}},\ x > 0$$

14
Solution

(a) $R_g = [0, \infty)$ $D_f = (-\infty, 2)$

For fg to exist, $R_g \subseteq D_f$

Since, $R_g \not\subset D_f$, fg does not exist.

For gf to exist, $R_f \subseteq D_g$

$R_f = (-1, \infty)$ and $D_g = (-\infty, \infty)$

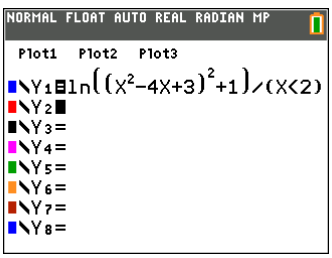
Since $R_f \subseteq D_g$, gf exists.

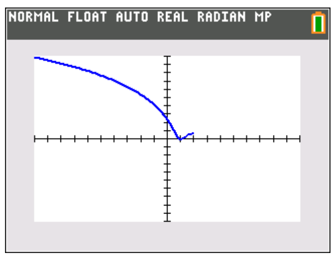
(b) $\text{gf}(x) = \text{g}(x^2 - 4x + 3)$

$$= \ln\left[(x^2 - 4x + 3)^2 + 1\right]$$

$D_{gf} = D_f = (-\infty, 2)$

Sketch the graph $\text{gf}(x) = \ln\left[(x^2 - 4x + 3)^2 + 1\right]$, $x < 2$.





From GC, $R_{gf} = (0, \infty)$

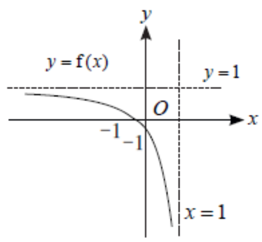
15
Solution

(a) Sketch the graph of function f to obtain the range of f.

We have $R_f = (-\infty, 1)$.

Since $R_f = (-\infty, 1) \not\subset D_g = (0, 1)$

Thus gf does not exist.



(b)
$$\begin{aligned} fg(x) &= f(\ln x) \\ &= 1 + \frac{2}{\ln x - 1} \\ &= \frac{\ln x + 1}{\ln x - 1} \end{aligned}$$ ◁ express as single fraction

$$\therefore \; fg(x) = \frac{\ln x + 1}{\ln x - 1}$$

There are two methods to find the value $(fg)^{-1}(0)$.

Method 1

Let $x = (fg)^{-1}(0)$ ◁ $y = f(x) \Leftrightarrow f^{-1}(y) = x$

$\therefore \; fg(x) = 0$

$$\frac{\ln x + 1}{\ln x - 1} = 0$$

$$\ln x + 1 = 0$$

$$\ln x = -1$$

$$x = e^{-1}$$

Since $x = (fg)^{-1}(0)$

$\therefore \; (fg)^{-1}(0) = e^{-1}$

Method 2

Let $y = fg(x)$

$$= \frac{\ln x + 1}{\ln x - 1}$$

$$y \ln x - y = \ln x + 1$$

$$\ln x (y - 1) = y + 1$$

$$\ln x = \frac{y+1}{y-1}$$

$$x = e^{\frac{y+1}{y-1}}$$

$$(fg)^{-1}(x) = e^{\frac{x+1}{x-1}}$$

$$\begin{aligned} \therefore \; (fg)^{-1}(0) &= e^{\frac{0+1}{0-1}} \\ &= e^{-1} \end{aligned}$$

(c)(i) Given $\quad \mathrm{gh}(x) = x$

$$\mathrm{h}(x) = \mathrm{g}^{-1}(x)$$

Since $\mathrm{g}^{-1}(x) = \mathrm{e}^{x}$

$\therefore \ \mathrm{h}(x) = \mathrm{e}^{x}$

(c)(ii) **Method 1**

Given $\quad \mathrm{hg}(x) = x^2 + 1 \qquad \triangleleft$ replace x by $\mathrm{g}^{-1}(x)$

$$\mathrm{hg}\left(\mathrm{g}^{-1}(x)\right) = \left(\mathrm{g}^{-1}(x)\right)^2 + 1$$

Since $\mathrm{g}^{-1}(x) = \mathrm{e}^{x}$ and $\mathrm{gg}^{-1}(x) = x$

$$\mathrm{h}(x) = (\mathrm{e}^{x})^2 + 1$$

$\therefore \ \mathrm{h}(x) = \mathrm{e}^{2x} + 1$

Method 2

Given $\quad \mathrm{hg}(x) = x^2 + 1$

$$\mathrm{h}[\mathrm{g}(x)] = x^2 + 1$$

$$\mathrm{h}[(\ln x))] = x^2 + 1$$

$$\mathrm{h}(\ln x) = (\mathrm{e}^{\ln x})^2 + 1 \qquad \triangleleft\ \mathrm{e}^{\ln x} = x$$

$$\mathrm{h}(x) = (\mathrm{e}^{x})^2 + 1 \qquad \triangleleft \text{ Replace } \ln x \text{ by } x$$

$\therefore \ \mathrm{h}(x) = \mathrm{e}^{2x} + 1$

16
Solution

(a) $D_f = [-\pi, \pi]$ and $R_f = [-1, 1]$

$D_g = [-1, \infty]$, $R_f = \left[-\frac{1}{4}, \infty\right)$

For gf to exists, $R_f \subseteq D_g$

$\therefore$ gf exists (Shown)

$$\text{gf}(x) = \text{g}[\text{f}(x)]$$
$$= 2\sin^2 x + \sin x$$

$\therefore$ $\text{gf} : x \mapsto 2\sin^2 x + \sin x, x \in \mathbb{R}, -\pi \le x \le \pi$

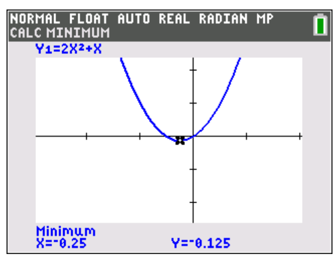
$$D_f = [-\pi, \pi] \xrightarrow{f} [-1, 1] \xrightarrow{g} \left[-\frac{1}{8}, 3\right]$$

$\therefore R_{gf} = \left[-\frac{1}{8}, 3\right]$

(b) Use GC to obtain the graph of $y = 2x^2 + x$.

From the graph, the minimum point $\left(-\frac{1}{4}, -\frac{1}{8}\right)$

Range of g $= \left[-\frac{1}{8}, \infty\right)$



First restricted domain of g such that g_1^{-1} exists,

$D_g = \left[-\frac{1}{4}, \infty\right]$ or $D_g = \left[-\infty, -\frac{1}{4}\right]$

For g_1 has the same range as g, the restricted domain to be $\left[-\frac{1}{4}, \infty\right]$.

$\therefore$ $D_{g_1} = \left[-\frac{1}{4}, \infty\right]$

17 Solution

(a) Let $y = \dfrac{ax+b}{cx-2}$

$$cxy - 2y = ax + b$$

$$x(cy-a) = b + 2y$$

$$x = \frac{b+2y}{cy-a}$$

$$= \frac{2y+b}{cy-a}$$

$$\therefore \ \mathrm{f}^{-1}(x) = \frac{2x+b}{cx-a}$$

Given that f is a self-inverse function, i.e. $\mathrm{f}(x) = \mathrm{f}^{-1}(x)$

$$\frac{ax+b}{cx-2} = \frac{2x+b}{cx-a}$$

Comparing the above equation.

$a = 2$ (Shown)

(b) Since f is a self-inverse function,

i.e. $\mathrm{f}(x) = \mathrm{f}^{-1}(x)$

$\mathrm{ff}(x) = \mathrm{ff}^{-1}(x)$ ◃ introduce function f on both sides

$$\mathrm{f}^2(x) = \mathrm{ff}^{-1}(x)$$

$$= x$$

$$\therefore \ \mathrm{f}^2(x) = x$$

(c) Already shown that f is a self-inverse function,

$\therefore$ $\mathrm{f}^n(x) = x$, if $n =$ even.

$\mathrm{f}^n(x) = \mathrm{f}(x)$, if $n =$ odd.

So, $\mathrm{f}^{71}(x) = \mathrm{f}(x)$

Hence, $\mathrm{f}^{71}(4) = \mathrm{f}(4)$

$$= \frac{2(4)+3}{5(4)-2}$$

$$\therefore \ \mathrm{f}^{71}(4) = \frac{11}{18}$$

(d) For fg to exist, $R_g \subseteq D_f$.

$R_g = [-3, \infty)$ and $D_f = \left(-\infty, \frac{2}{5}\right) \cup \left(\frac{2}{5}, \infty\right)$

Since $R_g \not\subset D_f$, fg does not exist.

(e)

$$\begin{aligned}
gf(x) &= 5 \\
g[f(x)] &= 5 \\
g\left(\frac{2x+3}{5x-2}\right) &= 5 \\
2\left(\frac{2x+3}{5x-2}\right)^2 - 3 &= 5 \\
2\left(\frac{2x+3}{5x-2}\right)^2 &= 8
\end{aligned}$$

$$\begin{aligned}
\left(\frac{2x+3}{5x-2}\right)^2 &= 4 \\
4x^2 + 12x + 9 &= 4(25x^2 - 20x + 4) \\
96x^2 - 92x + 7 &= 0 \\
x &= \frac{92 \pm \sqrt{8464 - 4(96)(7)}}{2(96)} \\
&= \frac{92 \pm 76}{192}
\end{aligned}$$

$\therefore\ x = \frac{7}{8}$ or $x = \frac{1}{12}$

18
Solution

(a) $a = 5$

5 has to be excluded from the domain of f as it does not have an image under f, which will then mean that f is not a function.

(b) Let $y = \dfrac{5x-3}{x-5}$.

$$\begin{aligned} y &= \frac{5x-3}{x-5} \\ y(x-5) &= 5x-3 \\ yx-5y &= 5x-3 \\ yx-5x &= 5y-3 \\ x &= \frac{5y-3}{y-5} \\ \mathrm{f}^{-1}(x) &= \frac{5x-3}{x-5} \end{aligned}$$

Since $\mathrm{f}^{-1}(x) = \mathrm{f}(x)$, $\therefore$ f is self-inverse.

(c) Since f is self-inverse, $\mathrm{f}^{-1}(x) = \mathrm{f}(x)$.

$$\begin{aligned} \mathrm{f}^4(x) &= \mathrm{ff}\left[\mathrm{f}^2(x)\right] \\ &= \mathrm{ff}\left[\mathrm{ff}^{-1}(x)\right] \\ &= \mathrm{ff}[x] \\ &= x \\ \mathrm{f}^4(b) &= b \end{aligned}$$

Given $\mathrm{f}^4(b) - 2 = \mathrm{f}^{-1}(b)$

$$\begin{aligned} b-2 &= \mathrm{f}(b) \\ b-2 &= \frac{5b-3}{b-5} \\ b^2-7b+10 &= 5b-3 \\ b^2-12b+13 &= 0 \end{aligned}$$

$\therefore\ b = 6+\sqrt{23}$ or $b = 6-\sqrt{23}$

(d) $$\left[\mathrm{e}^{10},\infty\right) \xrightarrow{\mathrm{g}} [10,\infty) \xrightarrow{\mathrm{f}} \left(5, \frac{47}{5}\right]$$

$$\mathrm{R}_{\mathrm{fg}} = \left(5, \frac{47}{5}\right]$$

Learning piont:

Take the $\mathrm{R}_{\mathrm{g}} = [10,\infty)$ as the restricted domain of f and read the corresponding range to obtain R_{fg}.

Refer to similar question: Follow up 14d.

19
Solution

(a) Use GC to graph function g and obtain range g. $\therefore\ \mathrm{R_g} = [-2, \infty)$.

For the composite function fg to exist, i.e. $\mathrm{R_g} \subseteq \mathrm{D_f}$.

As $[-2, \infty) \subseteq [-h, \infty)$

So $-2 \geq -h$

$\therefore\ h \geq 2$

(b) **Mapping Method**

$$\underbrace{(-1, \infty]}_{\mathrm{D_g}} \xrightarrow{\mathrm{g}} \underbrace{[-2, \infty)}_{\mathrm{R_g} = \text{Restricted } \mathrm{D_f}} \xrightarrow{\mathrm{f}} \underbrace{[\ln 3, \infty)}_{\mathrm{R_{fg}}}$$

Since $\mathrm{f}(-2) = \ln 3$

$\ln(-2 + 1 + h) = \ln 3$

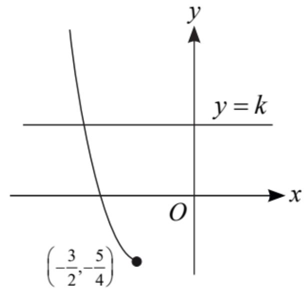
$-2 + 1 + h = 3$

$\therefore\ h = 4$

20
Solution

(a) Any horizontal line $y=k,(k\in\mathbb{R})$ cuts the graph of f at most once so f is a one- one function.

Hence, function f^{-1} exists.



(b) $R_g=(3,\infty)$ and $D_{f^{-1}}=R_f=\left[-\frac{5}{4},\infty\right)$

Since $R_g\subseteq D_{f^{-1}}$, $f^{-1}g$ exists.

$$y=x^2+3x+1$$
$$y=\left(x+\frac{3}{2}\right)^2+1-\frac{9}{4}$$
$$x=-\frac{3}{2}\pm\sqrt{y+\frac{5}{4}}$$

Since $x\le-\frac{3}{2}$, $x=-\frac{3}{2}-\sqrt{y+\frac{5}{4}}$

$$f^{-1}(x)=-\frac{3}{2}-\sqrt{x+\frac{5}{4}}$$
$$f^{-1}g(x)=f^{-1}\left(3+e^{-x}\right)$$
$$=-\frac{3}{2}-\sqrt{e^{-x}+\frac{17}{4}}$$

Domain of $f^{-1}g$, $D_{f^{-1}g}=D_g=\mathbb{R}$

(c) Given range of hg $=(\ln 5,\infty)$

R_g = Restricted domain h $=(3,\infty)$. $\quad\triangleleft$ R_g is obtained from the graph g

, $\quad h(3)=\ln 5$

$\ln(3-k+1)=\ln 5$

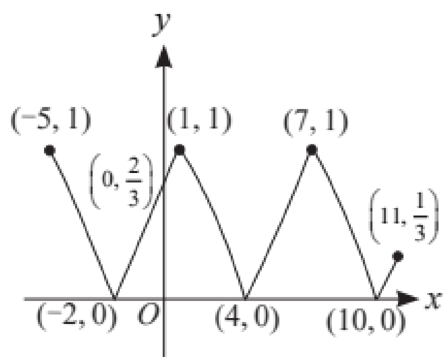
$\therefore\ k=-1$

Exercise 1

F Piecewise Functions

21
Solution

(a) The graph of $y = \mathrm{f}(x)$ for $-5 \le x \le 11$



(b) $\mathrm{f}(-10) = \mathrm{f}(2) = \sqrt{3} - 1$ and $\mathrm{f}(36) = \mathrm{f}(0) = \dfrac{2}{3}$

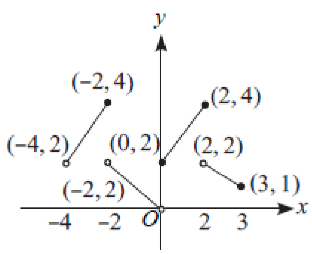
$\mathrm{f}(-10) + \mathrm{f}(36)$

$= \sqrt{3} - 1 + \dfrac{2}{3}$

$= \sqrt{3} - \dfrac{1}{3}$

22
Solution

(a) The graph of $y = \mathrm{f}(x)$ for $-4 < x \le 3$.



(b) **Method 1**

By the horizontal line test, the line $y = 2$ intersects the graph f more than once, f is not a 1 - 1 function.

Thus, f^{-1} does not exist.

Method 2

Since $\mathrm{f}(-2) = \mathrm{f}(2) = 4$, f is not 1 - 1 function.

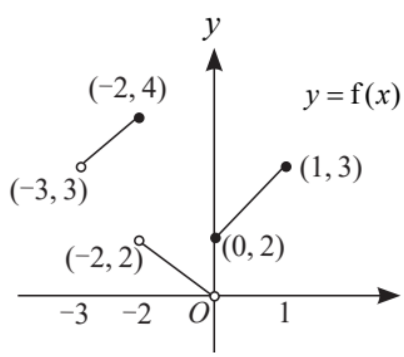
Thus, f^{-1} does not exist.

(c) For f^{-1} to exist, function f needs to be 1 - 1 function.

$\therefore\ a = -3$.

(d) Graph the function f with restricted domain $\mathrm{f} = (3, 1]$, as shown.

From the diagram, we note that



For $-3 < x \le -2$,

$\mathrm{D_f} = [-3, 2]$ and $\mathrm{R_f} = [3, 4]$

$y = x + 4 + 2$

$x = y - 6$

$\therefore\ \mathrm{f}^{-1}(x) = x - 6,\ x \in [3, 4]$

For $-2 < x < 0$,

$\mathrm{D_f} = [-2, 0]$ and $\mathrm{R_f} = [0, 2]$

$y = -x$

$x = -y$

$\therefore\ \mathrm{f}^{-1}(x) = -x,\ x \in [0, 2]$

For $0 \le x \le 1$,

$\mathrm{D_f} = [0, 1]$ and $\mathrm{R_f} = [2, 3]$

$y = x + 2$

$x = y - 2$

$\therefore\ \mathrm{f}^{-1}(x) = x - 2,\ x \in [2, 3]$

$$\therefore \quad f^{-1} : x \mapsto \begin{cases} x-6, & \text{for } 3 < x \le 4 \\ x-2, & \text{for } 2 \le x \le 3 \\ -x, & \text{for } 0 < x < 2 \end{cases}$$

23
Solution

(a) $f(0) = 1$

$f(1) = 2 - f(0) = 1$

$f(2) = 2f(1) = 2$

$f(3) = 2 - f(2) = 0$

$f(4) = 2f(2) = 4$

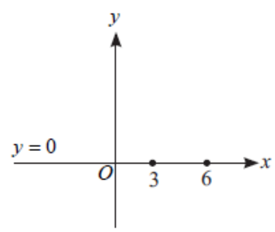
$f(5) = 2 - f(4) = -2$

$f(6) = 2f(3) = 0$

$\therefore$ $f(3) = 0$, $f(4) = 4$, and $f(6) = 0$

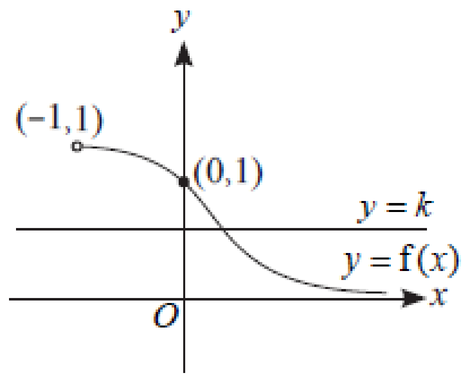
(b) Since $f(3) = f(6) = 0$ which give the same output value, therefore f is not a 1 - 1 function,
Hence, f does not have an inverse.

Alternative Method (Graphical)
Since $y = 0$ cuts the graph twice at $x = 3$ and $x = 6$, therefore f is not 1 - 1 function.
Hence, f does not have an inverse.



24
Solution

(a) The graph of $\mathrm{f}(x)=\mathrm{e}^{-(x+1)^2}$, for $x\in\mathbb{R}, x>-1$.



The line $y=k, k\in\mathbb{R}$, cuts the graph of $y=\mathrm{f}(x)$ at most once.
Therefore f is a one-one function. Hence f^{-1} exists.

Let $y=\mathrm{e}^{-(x+1)^2}$.

Then $\quad \ln y=-(x+1)^2$

$$x+1=\pm\sqrt{-\ln y}$$

$$x=-1\pm\sqrt{-\ln y}$$

Since $x>-1$, $x=-1+\sqrt{-\ln y}$

Hence $\mathrm{f}^{-1}: x\to -1+\sqrt{-\ln x},\quad 0<x<1$.

(b) $\mathrm{ff}^{-1}(x)=x$ when $x\in \mathrm{D}_{\mathrm{f}^{-1}}$.

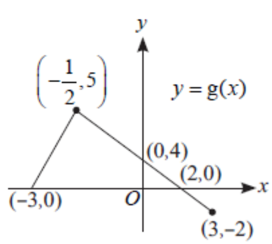
$\mathrm{D}_{\mathrm{f}^{-1}}=\mathrm{R}_{\mathrm{f}}=(0,1)$.

Hence $\mathrm{ff}^{-1}(x)=x$ when $x\in(0,1)$.

Range of values of x is $0<x<1$.

(c) Use GC to graph g.

From graph g, $\mathrm{R}_{\mathrm{g}}=[-2,5]$.



For fg to exist, $\mathrm{R}_{\mathrm{g}}\subseteq\mathrm{D}_{\mathrm{f}}$.

Now $\mathrm{R}_{\mathrm{g}}=[-2,5]$ and $\mathrm{D}_{\mathrm{f}}=[-1,\infty]$.

Since $[-2,5]\not\subset(-1,\infty)$, fg does not exist.

For gf to exist, $R_f \subseteq D_g$.
Now $R_f = (0,1)$ and $D_g = [-3,3]$.

Since $(0,1) \subset [-3,3]$, gf exists.

(d) $\mathrm{gf}(x) = \mathrm{g}\left(e^{-(x+1)^2}\right)$

$= 4 - 2e^{-(x+1)^2}, x \in (-1, \infty)$

$\mathrm{gf}(x) \to 4 - 2e^{-(x+1)^2}, \ x > 1$

Learning point:
Recall Rule: $D_{gf} = D_f$

$(-1, \infty) \xrightarrow{f} (0,1) \xrightarrow{g} (2, 4)$

$R_{gf} = (2, 4)$

25
Solution

(a) Consider the domain $\mathrm{f}:\ 0 \le x < a$

$$y = a\mathrm{e}^{a-x}$$
$$a - x = \ln\left(\frac{y}{a}\right)$$
$$x = a - \ln\left(\frac{y}{a}\right)$$
$$\mathrm{f}^{-1}(y) = a - \ln\left(\frac{y}{a}\right)$$
$$\mathrm{f}^{-1}(x) = a - \ln\left(\frac{x}{a}\right)$$

Learning point:

$$y = \mathrm{f}(x)$$
$$\mathrm{f}^{-1}(y) = x$$

For $\mathrm{D_f} = [0,\ a),\ \mathrm{R_f} = \mathrm{D_{f^{-1}}} = [a,\ a\mathrm{e}^{a})$

$\mathrm{f}^{-1}: x \to a - \ln\left(\frac{x}{a}\right),\quad a < x \le a\mathrm{e}^{a}$

Consider the domain $\mathrm{f}:\ a \le x \le 2a$,

$$y = a - \frac{1}{a}(x-a)^2$$
$$(x-a)^2 = a^2 - ay$$
$$x = a \pm \sqrt{a^2 - ay}$$

Since $x \ge a$,

$\therefore\ x = a + \sqrt{a^2 - ay}$ or $x = a - \sqrt{a^2 - ay}$ (Rejected)

$\therefore\ \mathrm{f}^{-1}(y) = a + \sqrt{a^2 - ay}$

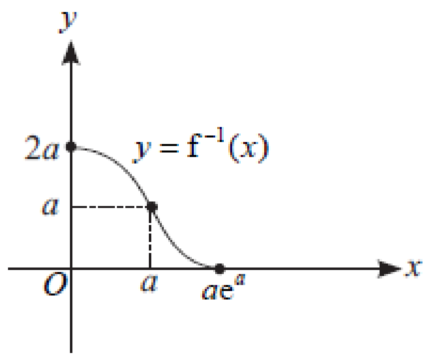
$\mathrm{f}^{-1}(x) = a + \sqrt{a^2 - ax}$

For $\mathrm{D_f} = [a,\ 2a],\ \mathrm{R_f} = \mathrm{D_{f^{-1}}} = [0,\ a]$

$\mathrm{f}^{-1}: x \to a - \ln\left(\frac{x}{a}\right),\quad a < x \le a\mathrm{e}^{a}$

$$\therefore\quad \mathrm{f}^{-1}: x \mapsto \begin{cases} a + \sqrt{a^2 - ax} & \text{for } 0 \le x \le a \\ a - \ln\left(\frac{x}{a}\right) & \text{for } a < x \le a\mathrm{e}^{a} \end{cases}$$

(b)



(c) $R_{f^{-1}} = [0, 2a]$ and $D_g = (0, \infty)$

$\because R_{f^{-1}} \not\subset D_g$

$\therefore gf^{-1}$ does not exist.

G Applications

26

Solution

(a) When $f(1) = 216$, it means that when the individual works 1 hour per day, his job satisfaction is 216.
When $f(9) = 216$, it means that the individual works 9 hours per day, his job satisfaction is 216.
Therefore, it is not necessarily true that when a person works long hours he will have high level of job satisfaction.

(b) $R_f = [75, 1200]$, $D_g = [0, 1000(e-1)]$

Since $R_f \subset D_g$, the composite function gf exist.

The composite function gf models the happiness index based on individual's job satisfaction.

(c)

$$[7,11] \xrightarrow{f} [208,416] \xrightarrow{g} [0.834, 0.920]$$

The range of values for gf is $[0.834, 0.920]$.

(d) Given f is an increasing function,
it implies that $f(b) > f(a)$ such that $b > a$.
and since g is a decreasing function,
it implies that $gf(b) > gf(a)$.

$\therefore$ the composite function gf will be a decreasing function.

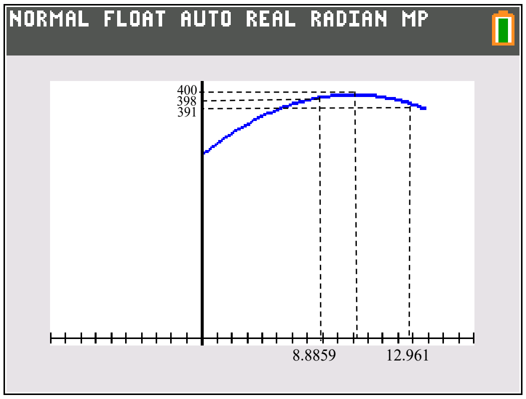
(e) Given that the happiness index based on individual's job satisfaction lies between 0.7 to 0.9 inclusive,
i.e. $0.7 \le gf(x) \le 0.9$

$$0.7 \le gf(x) \le 0.9$$
$$g^{-1}(0.7) \le f(x) \le g^{-1}(0.9)$$
$$258.68 \le f(x) \le 704.53$$
$$f^{-1}(258.68) \le x \le f^{-1}(704.53)$$
$$8.8859 \le x \le 12.961$$

$$[8.8859, 12.961] \xleftarrow{f^{-1}} [258.68, 704.53] \xleftarrow{g^{-1}} [0.7, 0.9]$$

$\therefore$ Range of the number of work hours $= [8.8859, 12.961]$.

Use GC to graph $h(x) = 400 - (x-10)^2$, for $8.8859 \le x \le 12.961$



When $x = 8.8859$, $h(8.8859) = 398$ (correct to nearest integer)
When $x = 12.961$, $h(12.961) = 391$ (correct to nearest integer)

From the graph, the range of the personal savings of an individual will be from 391 dollars to 400 dollars.

27
Solution

(a) When $t=0$, $\quad f(0)=25$

$$a\cos\left(\frac{\pi}{24}(0)-\frac{\pi}{2}\right)+b=25$$
$$b=25$$

When $t=12$, $\quad T=f(12)$

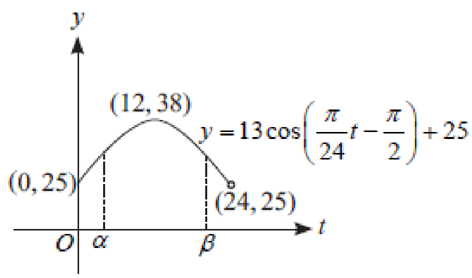
$$a\cos\left(\frac{\pi}{24}(12)-\frac{\pi}{2}\right)+b=38$$
$$a+b=38$$
$$a+25=38$$
$$a=13$$

(b) $\alpha\neq\beta$ but $f(\alpha)=f(\beta)$.

By symmetry of the curve.

$$\frac{\alpha+\beta}{2}=12$$

$\therefore \alpha+\beta=24$



Alternative method

$$\cos\left(\frac{\pi}{24}\alpha-\frac{\pi}{2}\right)=\cos\left(\frac{\pi}{24}\beta-\frac{\pi}{2}\right)$$
$$\left(\frac{\pi}{24}\alpha-\frac{\pi}{2}\right)=-\left(\frac{\pi}{24}\beta-\frac{\pi}{2}\right)$$

$\therefore \alpha+\beta=24$

(c) The composite function gf represents the rate of absorption of energy of a solar panel system t hours after 12 midnight on a typical day in May.

$R_f=[25,38]$, $D_g=(23,\infty)$

Since $R_f\subset D_g$, the function gf exists.

(d) From **(b)**

If $\alpha\neq\beta, \alpha+\beta=24$,

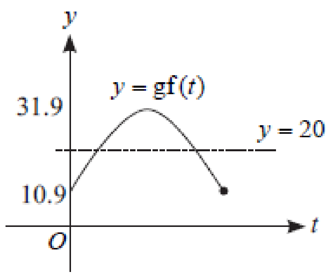
$f(\alpha)=f(\beta)$.

$\therefore gf(\alpha)=gf(\beta)$

Hence, gf is not one-one, $(gf)^{-1}$ does not exist.

Alternative method

Sketch the graph $y = \mathrm{gf}(t)$.



The line $y = 20$ cuts the graph of $y = \mathrm{gf}(t)$ more than once. gf is not one-one

Hence, $(\mathrm{gf})^{-1}$ does not exist.

(e) At s pm, $t = 12 + s$.

For $\qquad 12 - s < t < 12 + s$

$$13\cos\left(\frac{\pi}{24}(12-s)-\frac{\pi}{2}\right)+25 < \mathrm{f}(t) < 38$$

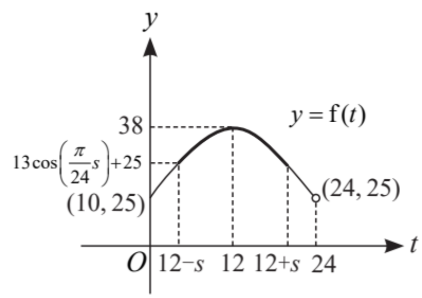
$$13\cos\left(\frac{\pi}{24}(12-s)-\frac{\pi}{2}\right)+25 < \mathrm{f}(t) < 38$$

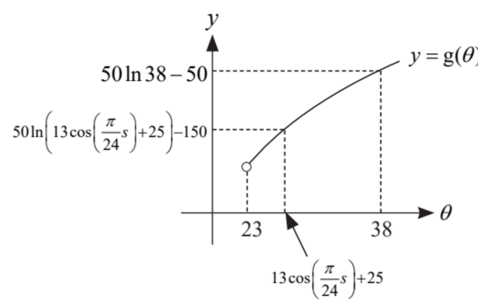
$$13\cos\left(\frac{12\pi}{24}-\frac{\pi}{24}s-\frac{\pi}{2}\right)+25 < \mathrm{f}(t) < 38$$

$$13\cos\left(-\frac{\pi}{24}s\right)+25 < \mathrm{f}(t) < 38 \qquad \because \cos\theta = \cos(-\theta)$$

$$13\cos\left(\frac{\pi}{24}s\right)+25 < \mathrm{f}(t) < 38$$

$$50\ln\left(13\cos\left(\frac{\pi}{24}s\right)+25\right)-150 < \mathrm{gf}(t) < 50\ln 38 - 150$$





$\therefore$ Range of rate of absorption

$$= \left(50\ln\left(13\cos\left(\frac{\pi}{24}s\right)+25\right)-150,\ 50\ln 38 - 150\right]$$

Alternative method

Graph the function $\mathrm{gf}(t) = 50\ln\left(13\cos\left(\frac{\pi}{24}t - \frac{\pi}{2}\right) + 25\right) - 150$.

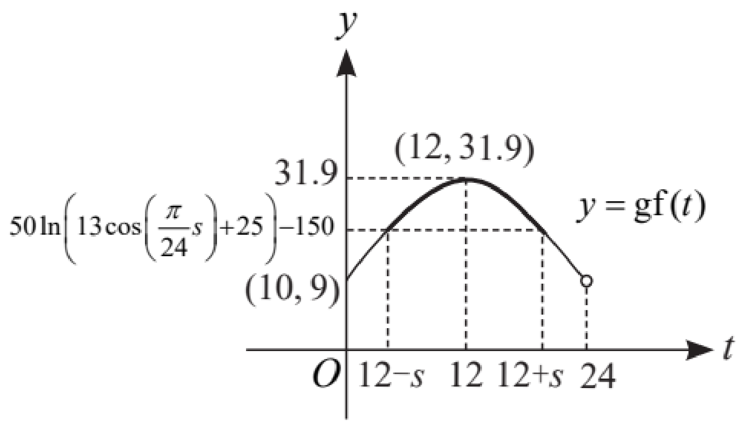
When $t = 12 - s$,

$$\mathrm{gf}(12-s) = 50\ln\left(13\cos\left(\frac{\pi}{24}(12-s) - \frac{\pi}{2}\right) + 25\right) - 150$$

$$= 50\ln\left(13\cos\left(\frac{\pi}{24}s\right) + 25\right) - 150$$

When $t = 12 + s$,

$$\mathrm{gf}(12+s) = 13\cos\left(\frac{\pi}{24}(12+s) - \frac{\pi}{2}\right) + 25$$



Since $12 - s + 12 + s = 24$, $\mathrm{gf}(12-s) = \mathrm{gf}(12+s)$.

Range of the rate of absorption of energy of a solar panel system

$$= \left(50\ln\left(13\cos\left(\frac{\pi}{24}s\right) + 25\right) - 150, 31.9\right] \quad \triangleleft \text{ (see graph gf)}$$

Exercise 1

H Mixed Practice

28
Solution

(a) $\mathrm{f}(x) = \sqrt{3}\sin x + \cos x$

$R\sin(x+a) = R\sin x\cos\alpha + R\cos x\sin\alpha$

$R\cos\alpha = \sqrt{3}$ .... (1)

$R\sin\alpha = 1$ .... (2)

$(1)^2 + (2)^2:\ \ R = \sqrt{(\sqrt{3})^2 + 1} = 2$

$\frac{(1)}{(2)}:\ \ \tan\alpha = \frac{1}{\sqrt{3}}$

$\therefore \qquad \alpha = \frac{\pi}{6}$

Hence $\mathrm{f}(x) = 2\sin\left(x + \frac{\pi}{6}\right)$

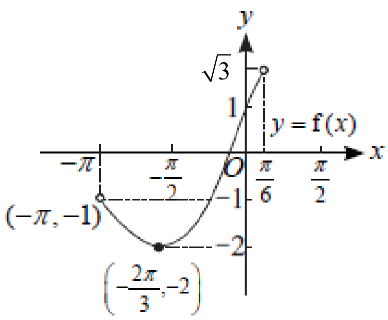
(b) When $y = -2$,

$2\sin\left(x + \frac{\pi}{6}\right) = -2$

$\sin\left(x + \frac{\pi}{6}\right) = -1$

$x + \frac{\pi}{6} = -\frac{\pi}{2}$

$x = -\frac{2\pi}{3}$



$\therefore$ turning point is $\left(-\frac{2\pi}{3}, -2\right)$.

From the graph, $\mathrm{R_f} = \left[-2, \sqrt{3}\right]$

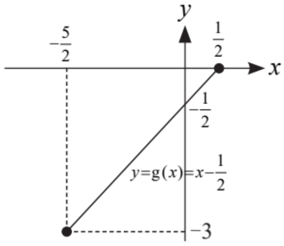
(c) From the graph above, the largest domain of f for f^{-1} exists is $\left[-\frac{2\pi}{3}, \frac{\pi}{6}\right)$.

Let $y = 2\sin\left(x + \frac{\pi}{6}\right)$

$x = \sin^{-1}\left(\frac{y}{2}\right) - \frac{\pi}{6}$

$\mathrm{f}^{-1} : x \mapsto \sin^{-1}\left(\frac{\pi}{2}\right) - \frac{\pi}{6},\ \ x \in \mathbb{R}, -2 \le x < \sqrt{3}$

(d) Graph function $g(x) = g : x \mapsto \frac{1}{2} - |x-1|, \; -\frac{5}{2} \le x \le \frac{1}{2}$



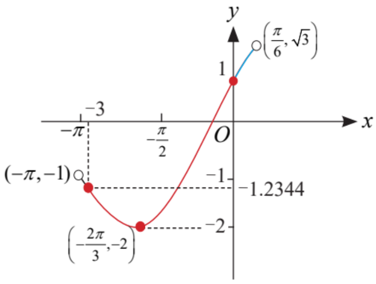
From the graph, $R_g = [-3, 0]$.

$R_g = [-3, 0]$ and $D_f = \left(-\pi, \frac{\pi}{6}\right)$

Since $R_g \subset D_f$, fg exists.

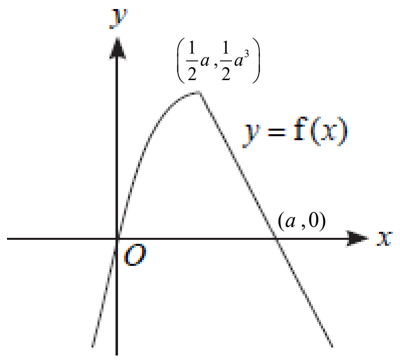
$$\overbrace{\left[-\frac{5}{2}, \frac{1}{2}\right]}^{D_g} \to \overbrace{[-3, 0]}^{R_g = \text{restricted } D_f} \to \overbrace{[-2, 1]}^{\text{restricted } R_f = R_{fg}}$$

From the graph on the right, $R_{fg} = [-2, 1]$.



29
Solution

(a) The graph of $y = \mathrm{f}(x)$



(b) When $x \le \frac{a}{2}$, every horizontal line $y = k,\ k \in \mathbb{R}_\mathrm{f}$ cuts the graph $y = \mathrm{f}(x)$ at exactly once.

Hence f is one - one and therefore f^{-1} exists.

The greatest value of k is $\frac{a}{2}$.

(c) Let

$$\begin{aligned} y &= \mathrm{f}(x) \\ &= a^2x - ax^2 \\ &= a\left(x - \frac{a}{2}\right)^2 + \frac{a^3}{4} \end{aligned}$$

$$\left(x - \frac{a}{2}\right)^2 = \frac{1}{a}\left(\frac{a^3}{4} - y\right)$$

$$x = \frac{a}{2} \pm \sqrt{\frac{1}{a}\left(\frac{a^3}{4} - y\right)}$$

Since $x \le \frac{a}{2}$, $\therefore x = \frac{a}{2} - \sqrt{\frac{1}{a}\left(\frac{a^3}{4} - y\right)}$

Hence, $\mathrm{f}^{-1} : x \to \frac{a}{2} - \sqrt{\frac{1}{a}\left(\frac{a^3}{4} - x\right)},\ x \in \mathbb{R}, x \le \frac{a^3}{4}$

The graph of $y = \mathrm{f}^{-1}(x)$ is a reflection of the graph of $y = \mathrm{f}(x)$ in the line $y = x$.

(d) $\mathrm{R}_\mathrm{f} = \left(-\infty, \frac{a^3}{4}\right]$ and $\mathrm{D}_\mathrm{g} = \left(-\infty, a^3\right]$.

Since $\mathrm{R}_\mathrm{f} \subseteq \mathrm{D}_\mathrm{g}$,

the composite function gf exists.

(e) $\mathrm{D}_\mathrm{f} \xrightarrow{\mathrm{f}} \mathrm{R}_\mathrm{f} \xrightarrow{\mathrm{g}} \left(-\infty, a^3\right]$

$\mathrm{D}_{\mathrm{gf}} = \left(-\infty, a^3\right]$

30
Solution

(a) Let $y = \dfrac{px}{x-1}$

$x = \dfrac{y}{y-p}$

$\therefore \mathrm{f}^{-1}(x) = \dfrac{x}{x-p}$

Since f is self-inverse, $p = 1$.

(b) $\mathrm{R_f} = (-\infty, 1) \cup (1, \infty)$ and $\mathrm{D_f} = (-\infty, 1) \cup (1, \infty)$

Since $\mathrm{R_f} = \mathrm{D_f}$, f^2 exists.

Learning point:

$\mathrm{R_f} = (-\infty, 1) \cup (1, \infty)$ written as $\mathrm{R_f} = \mathbb{R} \setminus \{1\}$
and $\mathrm{D_f} = (-\infty, 1) \cup (1, \infty)$ can be written as $\mathrm{D_f} = \mathbb{R} \setminus \{1\}$

(c) $\mathrm{f}^2(x) = \mathrm{ff}^{-1}(x)$

$= x$ for $x \in \mathbb{R}, x \neq 1$.

Range of f^2, $\mathrm{R_{f^2}} = (-\infty, 1) \cup (1, \infty)$

(d) Since $\mathrm{f}^{-1}(x) = \mathrm{f}(x)$,

$\therefore$ $\mathrm{f}^n(x) = x$, if $n =$ even.

$\mathrm{f}^n(x) = \mathrm{f}(x)$, if $n =$ odd.

So, $\mathrm{f}^{2019}(x) = \mathrm{f}(x)$

$\mathrm{f}^{2019}(x) = \dfrac{x}{x-1}$ for $x \in \mathbb{R}, x \neq 1$.

(e) Substitute the minimum point $\left(\dfrac{1}{4}, \ln\dfrac{39}{8}\right)$ into the function h.

$$\ln\frac{39}{8} = \ln\left(\frac{1}{16}a + \frac{1}{4}b + c\right)$$

$$\frac{1}{16}a + \frac{1}{4}b + c = \frac{39}{8}$$

$\therefore$ $a + 4b + 16c = 78$ .... (1)

$$\mathrm{h}'(x) = \frac{2ax+b}{ax^2+bx+c}$$

At $x = \dfrac{1}{4}$, $\mathrm{h}'(x) = 0$,

i.e. $$\frac{2a\left(\frac{1}{4}\right)+b}{a\left(\frac{1}{4}\right)^2 + +b\left(\frac{1}{4}\right)+c} = 0$$

$\therefore$ $a + 2b = 0$ .... (2)

$$\mathrm{gh}(x) = \mathrm{e}^{\ln(ax^2+bx+c)} + 2 \qquad \triangleleft\ \mathrm{e}^{\ln a} = a$$
$$= ax^2 + bx + c + 2$$

Given $\mathrm{gh}(1) = 8$,

i.e. $a(1)^2 + b(1) + c + 2 = 8$

$\therefore \qquad a + b + c = 6$ .... (3)

Using GC to solve (1), (2) and (3).

$\therefore\ a = 2, b = -1$ and $c = 5$.

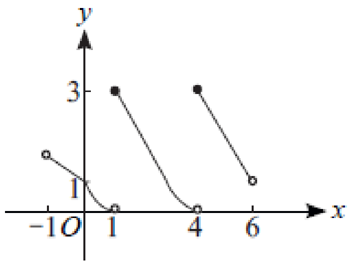
$\therefore\ \mathrm{h}(x) = \ln(2x^2 - x + 5)$

31
Solution

(a) When $x = 3$, $f(-3) = f(0)$.

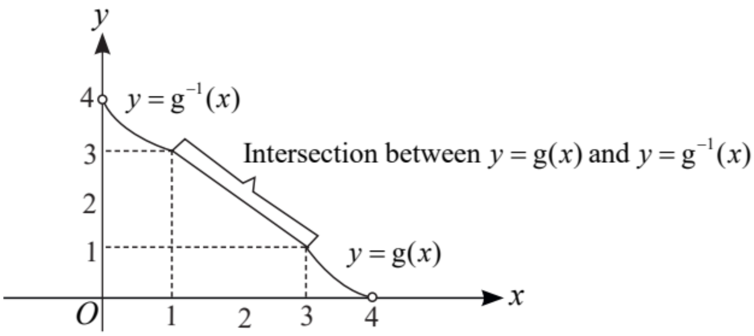
$\therefore$ f is not 1-1, hence f does not have an inverse.

(b) The graph of $y = f(x)$ for $-1 < x < 6$.



(c) $f(2017) = f(2014) = f(2011) = ... = f(1) = 3$

(d)



For the graph, the values of x such that $g(x) = g^{-1}(x)$ is $1 \le x \le 3$.

(e) $R_{g^{-1}} = [1,4) \not\subset [0,3] = D_h$

$\therefore hg^{-1}$ doesn't exist.

(f) For domain $1 \le x < 3$

$$\begin{aligned} hg(x) &= h[g(x)] \\ &= h[4-x] \\ &= (4-x-1)^4 \\ &= (3-x)^4 \end{aligned}$$

Learning point:

We know that for hg to exist, $R_g \subset D_g$.

R_g for $g(x) = 4-x$ is [1, 3], D_h for $h(x) = (x-1)^4$ is [1, 3] and D_h for $h(x) = \sqrt{1-x}$ is [0, 1)

We substitute $(4-x)$ into $g(x) = (x-1)^4$ since $R_g = [1, 3] \subset D_g = [1, 3]$.

For domain $3 \le x < 4$

$$\begin{aligned} hg(x) &= h[g(x)] \\ &= h[(x-4)^2] \\ &= \sqrt{1-(x-4)^2} \end{aligned}$$

Learning point:

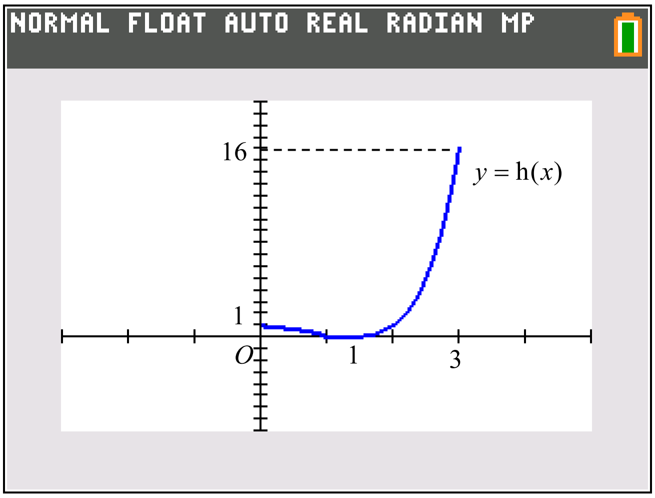
We know that for hg to exist, $R_g \subset D_g$.

R_g for $g(x) = (x-4)^2$ is (0, 1], D_h for $h(x) = (x-1)^4$ is [1, 3] and D_h for $h(x) = \sqrt{1-x}$ is [0, 1)

We substitute $(x-4)^2$ into $g(x) = \sqrt{1-x}$ since $R_g = (0, 1] \subset D_g = [0, 1)$.

$$hg(x) = \begin{cases} (3-x)^4 & \text{for } 1 \le x < 3, \\ \sqrt{1-(x-4)^2} & \text{for } 3 \le x < 4. \end{cases}$$

(g)



$$D_g = [1,4) \xrightarrow{g} R_g (\text{restricted } D_h) = (0,3] \xrightarrow{h} R_{hg} = [0,16]$$

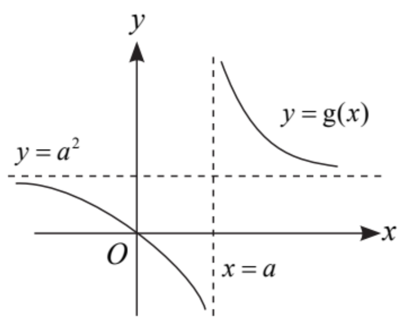
Therefore, range of hg $= [0,16]$

Exercise 1

I Higher Order Questions

32
Solution

(a) The graph of g



(b) From the graph of g in **(a)**, $R_g = \mathbb{R} \setminus \{a^2\}$

$D_{g^{-1}} = R_g = \mathbb{R} \setminus \{a^2\}$.

For $g^{-1}h$ to exist, $R_h \subseteq D_{g^{-1}}$.

$R_h = (a^2, \infty)$ and $D_{g^{-1}} = \mathbb{R} \setminus \{a^2\}$.

Since $R_h \subseteq D_{g^{-1}}$, $g^{-1}h$ exists.

(c) For the range of $g^{-1}h$,

$$D_h \xrightarrow{h} R_h = (a^2, \infty) \xrightarrow{g^{-1}} (a, \infty)$$

So, $R_{g^{-1}h} = (a, \infty)$

(d) Given $g^{-1}h(m) = m$

$$h(m) = g(m)$$

$$m^2 = \frac{a^2 m}{m-a}$$

$$m^3 - am^2 = a^2 m$$

$$m^3 - am^2 - a^2 m = 0$$

$$m(m^2 - am - a^2) = 0$$

$m = 0$ (rejected since $0 \notin D_h$) or $m^2 - am - a^2 = 0$ (Shown)

For $m^2 - am - a^2 = 0$

$$m = \frac{-(-a) \pm \sqrt{(-a)^2 - 4(1)(-a^2)}}{2(1)}$$

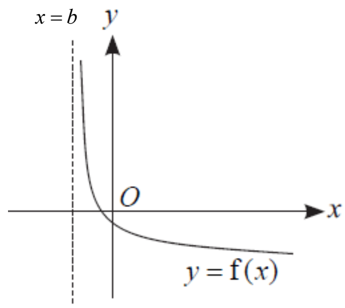
$$= \frac{a \pm \sqrt{5a^2}}{2}$$

$$= \frac{a}{2}(1 \pm \sqrt{5})$$

Since $m > a$, $m = \frac{a}{2}(1 + \sqrt{5})$.

33
Solution

(a) The graph of f



Since any horizontal line $y=k$ cuts the graph of f at most once, therefore f is one - one.

Let $y=-\ln(x-b)$

$x=\mathrm{e}^{-y}+b$

$\therefore\ \mathrm{f}^{-1}: x \mapsto \mathrm{e}^{-x}+b, x \in \mathbb{R}$

(b) $\mathrm{D}_{\mathrm{f}}=(b,\infty)$ and $\mathrm{R}_{\mathrm{g}}=[-1,\infty)$

For fg to exist, $\mathrm{R}_{\mathrm{g}} \subseteq \mathrm{D}_{\mathrm{f}}$

Since $[-1,\infty) \subseteq (b,\infty)$

Hence, greatest integer value of $b=-2$.

(c) $\mathrm{f}: x \mapsto -\ln(x+2),\ x>-2$

$$\mathrm{D}_{\mathrm{fg}}=\mathbb{R} \xrightarrow[\mathrm{g}]{} [-1,\infty) \xrightarrow[\mathrm{f}]{} (-\infty,0]$$

$\therefore$ the range of $\mathrm{fg}=(-\infty,0]$.

34
Solution

(a) Given $x < m < 2$, f is one-one function.

$\therefore$ f^{-1} exists.

$\mathrm{R_f} = \left(0,\ \dfrac{1}{(m-2)^2}\right)$ and $\mathrm{D_f} = (-\infty, m)$.

For f^2 to exist, $\mathrm{R_f} \subseteq \mathrm{D_f}$, the values of $\dfrac{1}{(m-2)^2}$ must lie between 0 and m.

i.e. $0 < \dfrac{1}{(m-2)^2} \le m$.

Consider $\dfrac{1}{(m-2)^2} \le m$.

$$1 \le m(m-2)^2$$
$$1 \le m(m^2 - 4m + 4)$$
$$m^3 - 4m^2 + 4m - 1 \ge 0$$

Solving with GC,

$0.382 \le m \le 1$ or $m \ge 2.84$ (reject since $m < 2$)

$\therefore$ $0.382 \le m \le 1$

(b)
$$y = \frac{1}{(x-2)^2}$$
$$x = 2 \pm \frac{1}{\sqrt{y}}$$

Since $x < m < 2$, $x = 2 - \dfrac{1}{\sqrt{y}}$.

$\mathrm{f}^{-1}(x) = 2 - \dfrac{1}{\sqrt{x}}$

$\mathrm{D}_{\mathrm{f}^{-1}} = \left(0, \dfrac{1}{(m-2)^2}\right)$ and $\mathrm{R}_{\mathrm{f}^{-1}} = (-\infty,\ m)$.

$\therefore$ $\mathrm{f}^{-1}(x) = 2 - \dfrac{1}{\sqrt{x}}$, $\mathrm{D}_{\mathrm{f}^{-1}} = \left(0, \dfrac{1}{(m-2)^2}\right)$, $\mathrm{R}_{\mathrm{f}^{-1}} = (-\infty,\ m)$.

$$f^2(x) = f\left(\frac{1}{(x-2)^2}\right)$$

$$= \frac{1}{\left(\frac{1}{(x-2)^2} - 2\right)^2}$$

$$= \frac{(x-2)^4}{\left(1-2(x-2)^2\right)^2}$$

$$f^2(x) = \frac{(x-2)^4}{\left(1-2(x-2)^2\right)^2}$$

$$D_f = D_{f^2} = (-\infty, m)$$

$$D_f = (-\infty, m) \xrightarrow{f} \left(0, \frac{1}{(m-2)^2}\right) \xrightarrow{f} \left(\frac{1}{4}, \frac{(m-2)^4}{\left(1-2(m-2)^2\right)^2}\right)$$

$$\text{Range of } f^2 = \left(\frac{1}{4}, \frac{(m-2)^4}{\left(1-2(m-2)^2\right)^2}\right)$$

$$\therefore\ f^2(x) = \frac{(x-2)^4}{\left(1-2(x-2)^2\right)^2},\ D_{f^2} = (-\infty, m),\ R_{f^2} = \left(\frac{1}{4}, \frac{(m-2)^4}{\left(1-2(m-2)^2\right)^2}\right)$$

35
Solution

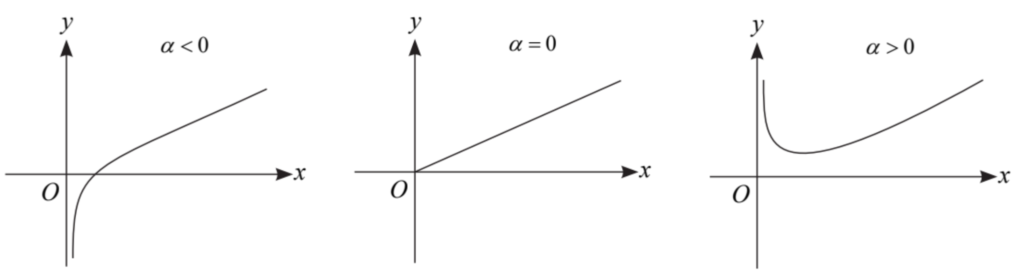
(a)(i) $\mathrm{g}(x) = x + \dfrac{\alpha}{x}$

$\mathrm{g}'(x) = 1 - \dfrac{\alpha}{x^2}$

If $\alpha > 0$, g will have a turning point at $x = \sqrt{\alpha}$, making g not an one-one function and g^{-1} will not exist.
If $\alpha = 0$, $\mathrm{g}(x) = x$ is a straight line. $\therefore$ g is a one-one function.
If $\alpha < 0$, $\mathrm{g}'(x) = 1 - \dfrac{\alpha}{x^2}$ is undefined. g is an increasing function and hence g is one-one function.

$\therefore$ range of values of α such that g^{-1} exists is $\alpha \leq 0$.

Learning point:
Consider the different scenarios :



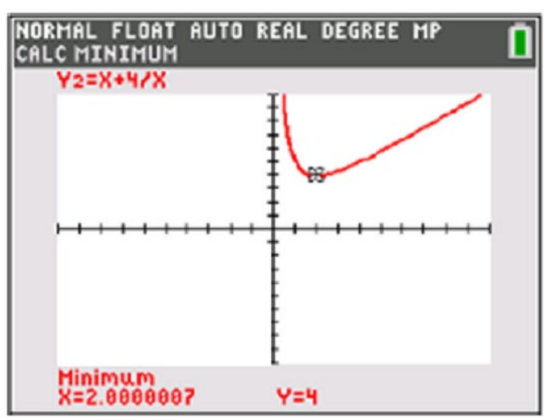
From the above diagrams, the range of values of α such that g^{-1} exists when $\alpha \leq 0$.

(ii) Given that $\alpha = 4$, $\mathrm{g}(x) = x + \dfrac{4}{x}$, $x > 0$

At stationary, $\mathrm{g}'(x) = 0$.

$$\therefore \quad 1 - \frac{4}{x^2} = 0$$

$$x = \pm 2$$



From the graph, $x \geq 2$, for g to be one-one function
$\therefore$ the least $\beta = 2$.

Let $y = \mathrm{g}(x)$

$\therefore \quad y = x + \frac{4}{x}$

$xy = x^2 + 4$

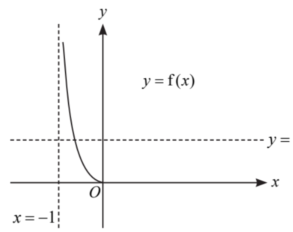
$x = \frac{y \pm \sqrt{y^2 - 4(1)(4)}}{2}$

Since $x \geq 2$, $\quad x = \frac{y + \sqrt{y^2 - 16}}{2}$

$\therefore \ \mathrm{g}^{-1}(x) = \frac{1}{2}(x + \sqrt{x^2 - 16})$

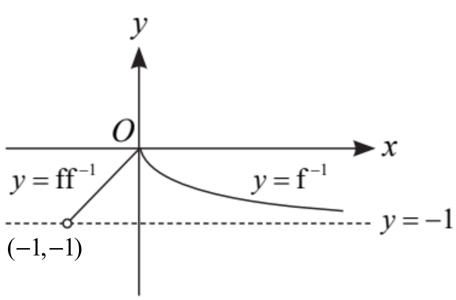
$\mathrm{D}_{\mathrm{g}^{-1}} = \mathrm{R}_{\mathrm{g}} = [4, \infty)$

(b)(i) From the diagram, when the domain f $= (-1, 0]$ the range of f remains the same as $= [0, \infty)$.



$\therefore$ restricted domain of f such that f^{-1} exists and the range of f remains unchanged $= (-1, 0]$.

(ii)



(iii) $\mathrm{f}^{-1}\mathrm{g}(x) = -\left(\frac{2 - \sqrt{1-x}}{3 - \sqrt{1-x}}\right)$

$= -\left(\frac{2 - \sqrt{1-x}}{1 + 2 - \sqrt{1-x}}\right)$

$\mathrm{f}^{-1}(\mathrm{g}(x)) = -\left(\frac{\mathrm{g}(x)}{\mathrm{g}(x) + 1}\right)$

$\mathrm{f}^{-1}(x) = -\frac{x}{x+1}$

$\therefore \ \mathrm{f}^{-1}(x) = -\frac{x}{x+1}$

Let $\quad y = -\dfrac{x}{x+1}$

$$x = -\frac{y}{y+1}$$

$\therefore\ \mathrm{f}(x) = -\dfrac{x}{x+1}$

$\mathrm{f} : x \to -\dfrac{x}{x+1},\ x > -1$

36
Solution

(a) For f^{-1} to exist, f must be 1-1 function. (See graph)

Determine the turning point by differentiation.

$\mathrm{f}(x)=\mathrm{e}^{(2x+\alpha)^2}$

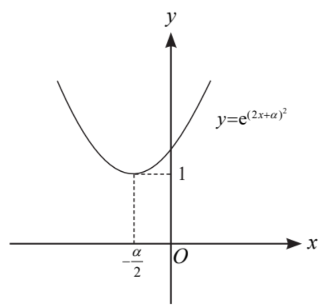
$\mathrm{f}'(x)=2(x+\alpha)\mathrm{e}^{(2x+\alpha)^2}$

At stationary, $\mathrm{f}'(x)=0$

$$2x+\alpha=0$$

$$x=\frac{\alpha}{2}$$

$\therefore\ k=-\dfrac{\alpha}{2}$



(b) Let $y=\mathrm{f}(x)$

$$y=\mathrm{e}^{(2x+\alpha)^2}$$

$$\ln y=(2x+\alpha)^2$$

$$2x+\alpha=\pm\sqrt{\ln y}$$

$$x=\frac{-\alpha\pm\sqrt{\ln y}}{2}$$

Since $x<-\dfrac{\alpha}{2}$, $\therefore\ \dfrac{-\alpha-\sqrt{\ln y}}{2}$

$\therefore\ \mathrm{f}^{-1}(x)=\dfrac{-\alpha-\sqrt{\ln x}}{2}$ and $\mathrm{D}_{\mathrm{f}^{-1}}=(1,\infty)$

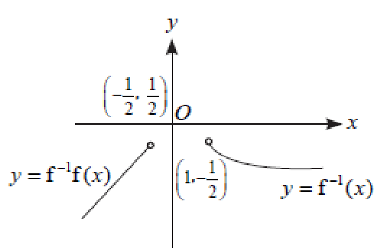
Learning point:

$\mathrm{D}_{\mathrm{f}^{-1}}=\mathrm{R}_{\mathrm{f}}(1,\infty)$ (refer to the diagram **(a)**)

(c) When $\alpha=1$, $\mathrm{f}(x)=\mathrm{e}^{(2x+1)^2}$.

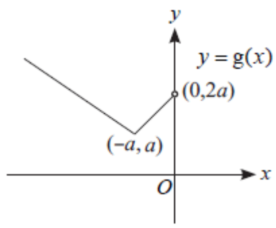
$\mathrm{D}_{\mathrm{f}}=\left(-\infty,-\dfrac{1}{2}\right]$ and $\mathrm{R}_{\mathrm{f}}=\left(-\dfrac{1}{2},1\right]$

Using GC to obtain the graph of f^{-1} and graph of $\mathrm{f}^{-1}\mathrm{f}$.



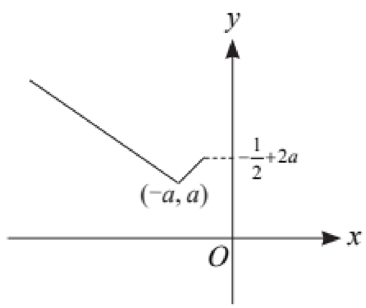
(d) $g(x) = |x+a| + a$

$$= \begin{cases} x+2a, & x \geq -a \\ -x, & x < -a \end{cases}$$



(e) $D_f = R_{f^{-1}} = \left(-\infty, -\frac{1}{2}\right)$ and $D_g = (-\infty,\ 0)$

Since $R_{f^{-1}} \subseteq D_g$, gf^{-1} exists. (Shown)



$$(1,\ \infty) \xrightarrow{f^{-1}} \left(-\infty, -\frac{1}{2}\right) \xrightarrow{g} [a,\ \infty)$$ ◁ see the above diagram

$R_{gf^{-1}} = [a,\ \infty)$

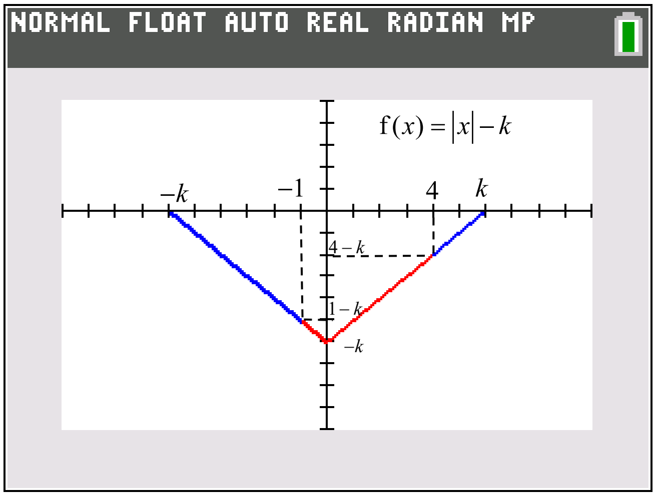
37
Solution

(a) $R_g = [-1, 4]$ and $D_f = (-2k, 2k)$, where $k > 2$

Since $R_g \subseteq D_f$. Thus fg exists.

(b)(i) $fg(-1) = f(0) = -k$

(ii)



$[-2,\ 2] \xrightarrow{g} [-1, 4] \xrightarrow{f} [-k, 4-k]$

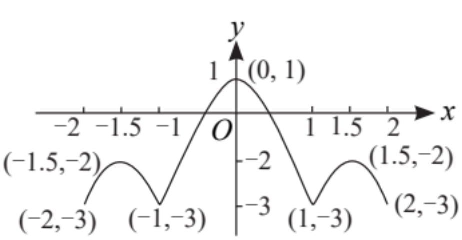
$R_{fg} = [-k, 4-k]$

Learning point:
Refer to the above diagram.
Using $R_g = [-1, 4]$ as the restricted D_f for the graph of $y = |x| - k$. (red curve) to obtain the range.
This range is the range of fg.
$\therefore\ R_{fg} = [-k, 4-k]$

(c) Given that $k = 3$, $f(x) = |x| - 3$, for $-6 < x < 6$.

$$fg(x) = f[g(x)]$$
$$= |\,g(x)\,| - 3$$

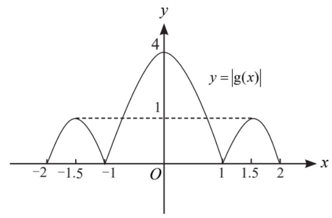
Graph of $y = fg(x)$



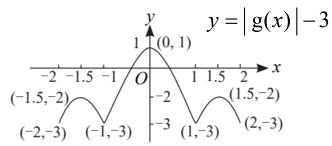
Learning point:

how to sketch $y = \mathrm{fg}(x)$

- Retains the graph above the x-axis and reflects the graph below x-axis in the x-axis.



- Translates the graph in the negative y-direction by 3 units. (i.e. move the graph down by 3 units)

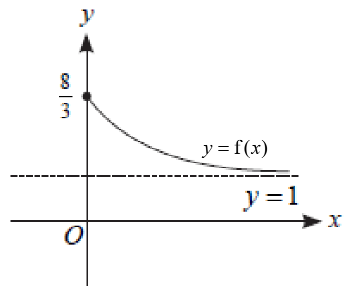


Exercise 1

J Exam Style Questions

38
Solution

(a) The graph of f



(b) Any horizontal line $y=k$, where $k\in\mathbb{R}$, cuts the graph of f at most once.
Hence, f is one-one. Hence f^{-1} exists.

Let $y=\mathrm{f}(x)$

$$y=1+\frac{5}{x^2+2x+3}$$

$$y-1=\frac{5}{x^2+2x+3}$$

$$y-1=\frac{5}{(x+1)^2+2}$$

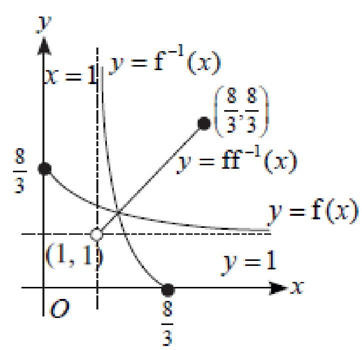
$$(x+1)^2=\frac{5}{y-1}-2$$

Since $x\geq 0$, $x=-1+\sqrt{\frac{5}{y-1}-2}$ and reject $x=-1-\sqrt{\frac{5}{y-1}-2}$

$$\therefore\ \mathrm{f}^{-1}(x)=-1+\sqrt{\frac{5}{x-1}-2}$$

$$\mathrm{D}_{\mathrm{f}^{-1}}=\mathrm{R}_{\mathrm{f}}=\left(1,\frac{8}{3}\right]$$

(c)



(d) $D_f = [0, \infty)$ and $R_f = \left(1, \frac{8}{3}\right]$

Since $R_f \subseteq D_f$, f^2 exists.

$D_f = [0, \infty) \xrightarrow{f} \left(1, \frac{8}{3}\right] \xrightarrow{f} \left[\frac{184}{139}, \frac{11}{6}\right)$

$\therefore$ the range of $f^2 = \left[\frac{184}{139}, \frac{11}{6}\right)$

Learning point:

When $x = 1$, $y = 1 + \frac{5}{(1)^2 + 2(1) + 3} = \frac{11}{6}$

When $x = \frac{8}{3}$, $y = 1 + \frac{5}{\left(\frac{8}{3}\right)^2 + 2\left(\frac{8}{3}\right) + 3} = \frac{184}{139}$

39
Solution

(a) Let $y=\mathrm{f}(x)=x^2+\lambda x-\lambda^2$

$$y=\left(x+\frac{\lambda}{2}\right)^2-\frac{5\lambda^2}{4}$$

$$x=-\frac{\lambda}{2}-\sqrt{y+\frac{5\lambda^2}{4}} \quad \text{or} \quad x=-\frac{\lambda}{2}+\sqrt{y+\frac{5\lambda^2}{4}} \text{ (rejected since } x\le-\frac{\lambda}{2}\text{)}$$

$$\therefore\ \mathrm{f}^{-1}(x)=-\frac{\lambda}{2}-\sqrt{x+\frac{5\lambda^2}{4}}$$

$$\mathrm{D}_{\mathrm{f}^{-1}}=\left[-\frac{5\lambda^2}{4},\infty\right) \text{ and } \mathrm{R}_{\mathrm{f}^{-1}}=\left(-\infty,\frac{-\lambda}{2}\right]$$

Alternative Method (to find f inverse)

$y=x^2+\lambda x-\lambda^2$

$x^2+\lambda x-\lambda^2-y=0$

Using quadratic formula,

$$x=\frac{-\lambda\pm\sqrt{\lambda^2-4(1)(-\lambda^2-y)}}{2}$$

$$x=-\frac{\lambda}{2}+\frac{\sqrt{5\lambda^2+4y}}{2} \text{ (rejected since } x\le-\frac{\lambda}{2}\text{)} \quad \text{or} \quad x=-\frac{\lambda}{2}-\frac{\sqrt{5\lambda^2+4y}}{2}$$

$$\mathrm{f}^{-1}(x)=-\frac{\lambda}{2}-\frac{\sqrt{5\lambda^2+4x}}{2}$$

(b) $\mathrm{R_g}=(0,\infty)$ and $\mathrm{D_f}=\left(-\infty,-\frac{\lambda}{2}\right]$

Since $\mathrm{R_g}\not\subset\mathrm{D_f}$, fg does not exist.

(c)
$$\begin{aligned}\mathrm{gf}(x)&=\mathrm{g}(x^2+\lambda x-\lambda^2)\\&=\mathrm{e}^{x^2+\lambda x-\lambda^2}\end{aligned}$$

$$\therefore\ \mathrm{gf}:x\mapsto\mathrm{e}^{x^2+\lambda x-\lambda^2},\ x\le-\frac{\lambda}{2}$$

Learning point:

$$\mathrm{D_{gf}}=\mathrm{D_f}=\left(-\infty,-\frac{\lambda}{2}\right]$$

$$\left(-\infty,-\frac{\lambda}{2}\right]\xrightarrow{\mathrm{f}}\left[-\frac{5\lambda^2}{4},\infty\right)\xrightarrow{\mathrm{g}}\left[\mathrm{e}^{-\frac{5\lambda^2}{4}},\infty\right)$$

The range of gf, $\mathrm{R_{gf}}=\left[\mathrm{e}^{-\frac{5\lambda^2}{4}},\infty\right)$

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Solution

(a) $\mathrm{ff}(x)=\dfrac{\dfrac{x+a}{x+b}+a}{\dfrac{x+a}{x+b}+b}$

$=\dfrac{x+a+a(x+b)}{x+a+b(x+b)}$

$=\dfrac{x+ax+a+ab}{x+bx+a+b^2}$

Given $\quad \mathrm{ff}(x)=\mathrm{g}(x)$

$\dfrac{x+ax+a+ab}{x+bx+a+b^2}=x$

$x+ax+a+ab=x(x+bx+a+b^2)$

$=x^2+bx+ax+b^2x$

$(1+b)x^2+(b^2-1)x=a(1+b)$

Compare the coefficients of x^2

$1+b=0$

$\therefore b=-1$

(b) Let $\quad y=\dfrac{x+a}{x-1}$

$y(x-1)=x+a$

$xy-y=x+a$

$xy-x=y+a$

$x=\dfrac{y+a}{y-1}$

$\therefore\ \mathrm{f}^{-1}(x)=\dfrac{x+a}{x-1}$

(c) The condition is $a=b$.

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Solution

(a) Given $\mathrm{f}: x \mapsto -2x^2 + 4x,\ x \in \{-2,-1,0,1,2\}$

$\mathrm{D_f} = \{-2,-1,0,1,2\}$

$\mathrm{R_f} = \{-16,-6,0,2\}$

Since $\mathrm{f}(0) = \mathrm{f}(2) = 0$, it is not a 1-1 function.

$\therefore$ f does not have an inverse.

(b) $\mathrm{R_f} = \{-16,-6,0,2\}$ and $\mathrm{D_g} = (-\infty, a) \cup (a, \infty)$

For gf does not exist, $\mathrm{R_f} \not\subset \mathrm{D_g}$.

$\therefore\ a = 2$

(c) Let
$$y = \frac{ax}{x-a}$$
$$yx - ya = ax$$
$$yx - ax = ya$$
$$x(y-a) = ya$$
$$x = \frac{ya}{y-a}$$
$$\mathrm{g}^{-1}(y) = \frac{ya}{y-a}$$
$$\mathrm{g}^{-1}(x) = \frac{ax}{x-a},\ x \neq a$$

$\mathrm{g}(x) = \mathrm{g}^{-1}(x)$ for all $x,\ x \neq a$ (Shown)

(d) Since $\mathrm{g}(x) = \mathrm{g}^{-1}(x)$

$\mathrm{g}^2(x) = x$

$\mathrm{g}^3(x) = \mathrm{g}(\mathrm{g}^2(x)) = \mathrm{g}(x)$

$$\begin{aligned}\mathrm{g}^{2019}\mathrm{f}(1) &= \mathrm{g}^{2019}(2)\\ &= \mathrm{g}(2)\\ &= \frac{3(2)}{2-3}\\ &= -6\end{aligned}$$

(d) $g^{2019}f(1) = g^{2019}(2)$

In **(c)**, g is self-inverse, Since $g(x) = g^{-1}(x)$

$\therefore \quad g^{n}(x) = x\,, \qquad$ if $n =$ even.

$\quad\;\; g^{n}(x) = g(x)\,, \quad$ if $n =$ odd.

So, $g^{2019}(x) = g(x)$.

$$
\begin{aligned}
g^{2019}(2) &= g(2) \\
&= \frac{3(2)}{2-3} \\
&= -6
\end{aligned}
$$

$\therefore\; g^{2019}(2) = -6$

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Solution

(a) **(i)** Let $y = \mathrm{f}(x)$.

$$y = \frac{3x+k}{x-b}$$
$$xy - by = 3x + k$$
$$x(y-3) = by + k$$
$$x = \frac{by+k}{y-3}$$
$$\mathrm{f}^{-1}(y) = \frac{by+k}{y-3}$$
$$\therefore\ \mathrm{f}^{-1}(x) = \frac{bx+k}{x-3}$$

Given f is self-inverse, i.e. $\mathrm{f}(x) = \mathrm{f}^{-1}(x)$.

$$\therefore \quad \frac{3x+k}{x-b} = \frac{bx+k}{x-3}$$

By observing the denominators.

$\therefore\ b = 3$

When $b = 3$, now $\dfrac{3x+k}{x-3} = \dfrac{3x+k}{x-3}$ $\quad \triangleleft$ replace $b = 3$

If $k = -9$, then function f will become a horizontal line. Hence f^{-1} does not exist.

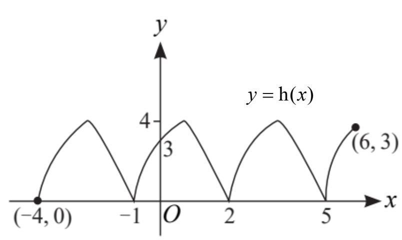
$\therefore\ k \in \mathbb{R},\ k \neq -9$

(ii) $\mathrm{fg}(x) = 2x - 1$

$$\begin{aligned} \mathrm{g}(x) &= \mathrm{f}^{-1}(2x-1) \\ &= \mathrm{f}(2x-1) \\ &= \frac{3(2x-1)+k}{(2x-1)-3} \\ &= \frac{6x-3+k}{2x-4},\ x \neq 2 \end{aligned}$$

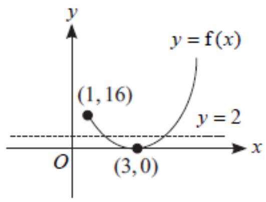
$$\therefore\ \mathrm{g}(x) = \frac{6x-3+k}{2x-4},\ x \neq 2$$

(b) The graph $y = \mathrm{h}(x)$



43
Solution

(a) The graph $y = \mathrm{f}(x)$



From the graph $y = \mathrm{f}(x)$, a horizontal line $y = 2$ cuts the curve at 2 points, so f is <u>not one - one</u>.

Hence f does not have an inverse. (Shown)

Alternative

From graph of $y = \mathrm{f}(x)$, a horizontal line $y = k$, where $0 < k \le 16$, cuts the curve at 2 points, hence f is <u>not one - one</u>.

Hence f <u>does not have an inverse</u>. (Shown).

(b) For function f inverse to exist, function f needs to be 1 - 1.

This occurs when $x \ge 3$. ◃ (see diagram in **(a)**)

$\therefore$ the minimum value of a is $a = 3$.

Letx $\quad y = (x-3)^4$

$x - 3 = \pm\sqrt[4]{y}$

$x = 3 + \sqrt[4]{y}$ or $x = 3 - \sqrt[4]{y}$ (reject since $x \ge 3$)

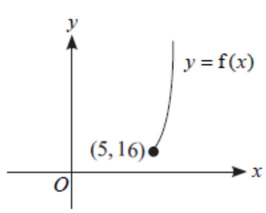
$\therefore\ \mathrm{f}^{-1}(x) = 3 + \sqrt[4]{x}$

$\mathrm{f}^{-1} : x \mapsto 3 + \sqrt[4]{x}$, for $x \in \mathbb{R}, x \ge 0$.

(c) Graph the function f to obtain the range of f, as shown in the diagram.

$\mathrm{R_f} = [16, \infty)$ and $\mathrm{D_g} = (3, \infty)$

Since $\mathrm{R_f} \subset \mathrm{D_g}$, gf exists.



$$\mathrm{gf}(x) = \mathrm{g}\left((x-3)^4\right), x \ge 5$$
$$= \ln\left((x-3)^4 - 3\right)$$

Given $\mathrm{gf}(x) = \ln 78$

$$\ln\left((x-3)^4 - 3\right) = \ln 78$$
$$(x-3)^4 = 81$$
$$x - 3 = 3 \quad \text{or} \quad x - 3 = -3$$
$$x = 6 \ \ \text{(within domain of gf)} \qquad x = 0 \ \ (\text{reject since } x \ge 5)$$

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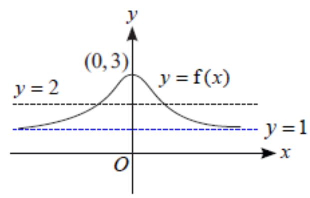
Solution

(a) **Method 1 : Horizontal Line Test**

$f: x \mapsto 1 + 2e^{-x^2}, x \in \mathbb{R}$



Since the horizontal line $y = 2$ that intersects the graph of $y = f(x)$ more than once, f is not one-one.

f does not have an inverse.

Method 2 : Counterexample

Since $f(-1) = f(1) = 1 + \frac{2}{e}$, f is **not one-one.**

f does not have an inverse.

(b) Largest value of k is 0. $\triangleleft$ (refer to the graph above)

(c) Let $y = 1 + 2e^{-x^2}$

$$y - 1 = 2e^{-x^2}$$
$$e^{-x^2} = \frac{y-1}{2}$$
$$-x^2 = \ln\left(\frac{y-1}{2}\right)$$
$$x^2 = -\ln\left(\frac{y-1}{2}\right)$$
$$x^2 = \pm\sqrt{-\ln\left(\frac{y-1}{2}\right)}$$

Since $x \le 0$ (restricted domain of f), $x = -\sqrt{-\ln\left(\frac{y-1}{2}\right)}$

$\therefore\ f^{-1}(x) = -\sqrt{-\ln\left(\frac{x-1}{2}\right)}$

(d) Range of f $= (1, 3]$ and Domain of g = Range of $g^{-1} = (1, 3]$

Since Range of f $\subseteq$ Domain of g,

gf exists.

(e) Since $R_f = D_g$, Range of gf = Range of f $= (-\infty, 0]$

Range of gf $= (-\infty, 0]$

Domain of gf $= \mathbb{R}$

Learning point:

For gf exists, $R_f \subseteq D_g$ and furthermore $R_{gf} \subseteq R_f$.

If $R_f = D_g$, then $R_{gf} = R_f$

(f) Given that $\mathrm{gf}(x) = x$,

so $\mathrm{gf}(-2) = -2$

$$\mathrm{g}^{-1}\mathrm{gf}(-2) = \mathrm{g}^{-1}(-2) \qquad \triangleleft\ \mathrm{g}^{-1} \text{ both sides}$$

$$\mathrm{g}^{-1}(-2) = \mathrm{f}(-2)$$

$$= 1 + 2\mathrm{e}^{-4}$$

$\therefore\ \mathrm{g}^{-1}(-2) = 1 + 2\mathrm{e}^{-4}$

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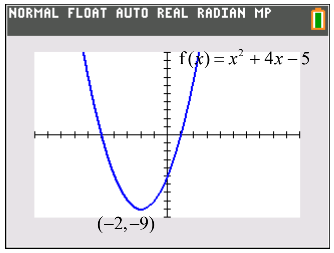
Solution

(a) Refer to the graph $f(x) = x^2 + 4x - 5$



For f to be 1-1 function, $x \geq -2$ or $x \leq -2$.

Largest exact value of $k = -2$.

Let $y = f(x)$

$$y = x^2 + 4x - 5$$
$$= (x+2)^2 - 9$$
$$x + 2 = \pm\sqrt{y+9}$$
$$x = -2 \pm \sqrt{y+9}$$

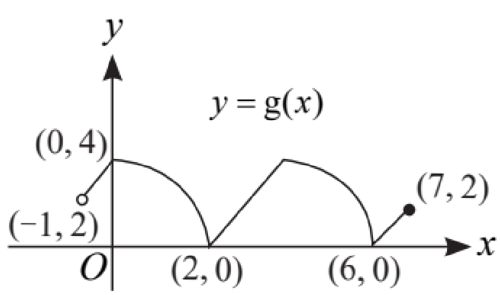
Since $x \leq -2$, $\quad x = -2 - \sqrt{y+9}$ and $x = -2 + \sqrt{y+9}$ is rejected.

$\therefore \quad x = -2 - \sqrt{x+9}$

Since $D_{f^{-1}} = R_f = [-9, \infty)$,

$f^{-1} : x \mapsto -2 - \sqrt{x+9}$, for $x \geq -9$

(b) The graph of $y = g(x)$ for $-1 < x \leq 7$.



(c) $R_g = [0, 4]$ from part **(b)** and $D_{f^{-1}} = [-9, \infty)$ from part **(a)**

Since $R_g \subseteq D_{f^{-1}}$, $f^{-1}g$ exists.

$$f^{-1}g(6) = f^{-1}[g(6)]$$
$$= f^{-1}[g(2)] \quad \triangleleft \text{ since } g(x) = g(x+4), \ \therefore \ g(2) = g(6)$$
$$= f^{-1}[2(2) - 4] \quad \triangleleft \text{ subsitute } x = 2 \text{ into } g(x) = 2x - 4$$
$$= f^{-1}(0)$$
$$= -2 - \sqrt{9}$$
$$= -5$$

$\therefore \ f^{-1}g(6) = -5$